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STABILITY OF RIGID MOTIONS AND ROLLERS INBICOMPONENT  
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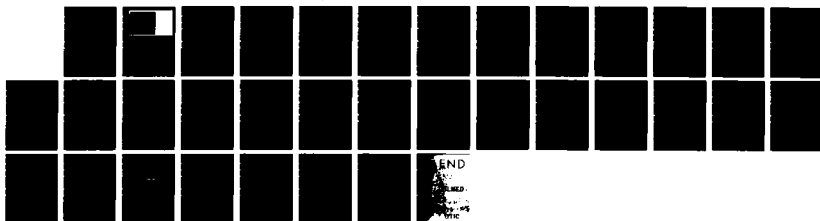
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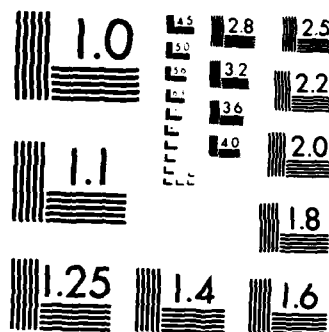
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STABILITY OF RIGID MOTIONS AND  
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IMMISCIBLE LIQUIDS

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STABILITY OF RIGID MOTIONS AND ROLLERS IN  
BICOMPONENT FLOWS OF IMMISCIBLE LIQUIDS

Daniel D. Joseph<sup>\*,1</sup>, Y. Renardy<sup>2</sup>, M. Renardy<sup>2</sup>, K. Nguyen<sup>\*\*</sup>

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ABSTRACT

We consider the motion of two rings of liquids with different viscosities and densities lying between concentric cylinders that rotate with the same angular velocity  $\Omega$ . Gravity is neglected and interfacial tension is included. We show that rigid motions are globally stable and that the shape of the interface which separates the two fluids is determined by a minimizing problem for a potential  $P$  defined as the negative of the sum of the kinetic energies of two rigid motions plus the surface energy of the interface. We show that the stable interface between fluids has a constant radius when heavy fluid is outside and  $J$  is larger than one, where  $J = -d^3[[\rho]]\Omega^2/T$  where  $d$  is the mean radius,  $[[\rho]] < 0$  the density difference and  $T$  the surface tension. When  $J$  is negative the heavy fluid is inside and the interface must be corrugated. The potential of flows with heavy fluid outside is smaller, thus relatively more stable, than when light fluid is outside, whenever  $J$  is large or for any  $J$  when the volume ratio  $m$  of heavy to light fluid is greater than one. These results give partial explanation of the stability and shape of rollers of viscous oils rotating in water and the corrugation of the free surface of films coating rotating cylinders.

AMS(MOS) Classifications: 76E99, 76T05, 76U05.

Key Words: Hydrodynamic Stability, Two Component Flow, Centrifuge.

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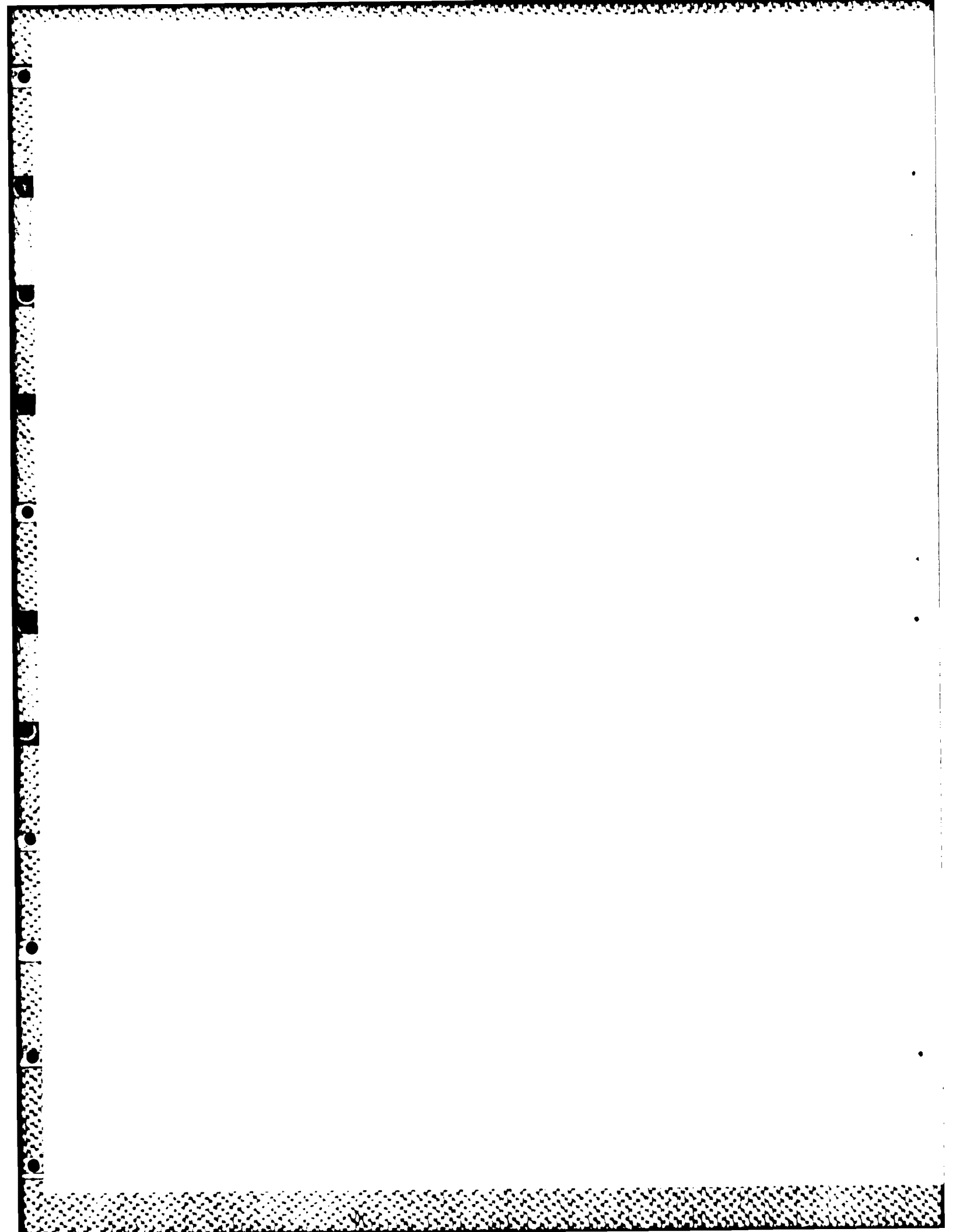
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## SIGNIFICANCE AND EXPLANATION

The authors study the stability of flows of two immiscible fluids between two cylinders rotating at the same frequency. They show that this problem can be reduced to the minimization of a certain functional. Necessary and sufficient conditions for the stability of "centrifuged" configurations with the heavier fluid on the outside, are given. It is shown that the viscosity plays no role in the stability problem. This is different from shearing flows between rotating cylinders, where it is in fact possible to have the heavier fluid inside.



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The responsibility for the wording and views expressed in this descriptive summary lies with MRC and not with the authors of this report.



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1. INTRODUCTION

We consider the flow of two immiscible liquids with different viscosities and densities lying between concentric cylinders both of which rotate with the same angular velocity  $\Omega$ . We neglect gravity and include interfacial tension. We study the stability of steady rigid-body rotation in which the two fluids are arranged in two rings with a given volume ratio. We show that rigid-body rotation is globally stable, and the interface shape between the two fluids is determined by a minimizing problem for a potential defined as the negative of the sum of the kinetic energies of two rigid motions plus the surface energy of the interface. We show that the interface between the two fluids has a constant radius when the heavy fluid is outside and  $J$  is larger than one, where  $J$  is defined in the abstract. This implies that centrifuged configurations lie outside an interface of constant radius. We note that the stable rollers of oil in water observed by Joseph, Nguyen and Beavers (1984) have heavy fluid (water) outside an interface of essentially constant radius. The rollers are maintained in nearly rigid motion by the high viscosity; there is no outer cylinder, and the motion of the water is not rigid. We show that the interface on rigid motions with heavy fluid inside must be corrugated. Photographs of corrugated interfaces of liquid films coating cylinders rotating in air can be found in the papers of Yih (1960) and Moffatt (1977). Yih (1960) gave a linear stability analysis for films coating cylinders rotating in air and his results are consistent with ours.

The potential of flows with heavy fluid outside is smaller, thus relatively more stable, than when light fluid is outside, when  $J$  is large or for any  $J$  when the volume ratio  $m$  of heavy to light fluid is greater than one. This is consistent with the idea that configurations with heavier fluid outside should be more stable because of the centrifugal force and that if the inner fluid is heavier, the rigid motion should be less stable no matter what the

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viscosities. The stability of flows in which viscosity differences are important depend strongly on the viscosity ratio. Rigid-body rotation involves no shear. As a result, the mechanism which we call lubrication stabilization (Renardy & Joseph, 1984), in which thin layers of the less viscous fluid occupy regions of high shear, is absent.

## 2. EQUATIONS OF MOTION AND INTERFACE CONDITIONS

Consider the flow of two immiscible liquids contained between two infinite concentric cylinders. The perturbed regions occupied by liquid 1 and 2 are denoted respectively as

$$V_1(t) = \{r, \theta, x | a \leq r \leq R(x, \theta, t), -\infty < x < \infty, 0 \leq \theta \leq 2\pi\},$$

$$V_2(t) = \{r, \theta, x | R(x, \theta, t) \leq r \leq b, -\infty < x < \infty, 0 \leq \theta \leq 2\pi\}.$$

The stress is given by

$$\underline{T} = -p\underline{I} + \underline{S}, \quad \underline{S} = 2\mu \underline{D}[\underline{u}]. \quad (2.1)$$

The equations of motion hold in each region

$$\operatorname{div} \underline{u}_l = 0, \quad l = 1, 2, \quad (2.2)$$

$$\rho_l \frac{d}{dt} \underline{u}_l = -\nabla \phi + \operatorname{div} \underline{S}_l, \quad (2.3)$$

where  $\underline{u}_l = \underline{e}_r u_l + \underline{e}_\theta v_l + \underline{e}_x w_l$ ,  $\phi = p + \rho g r \sin \theta$ ,  $\underline{D}[\underline{u}] = \frac{1}{2}(\nabla \underline{u} + \nabla \underline{u}^T)$ ,  $\rho_1$  and  $\rho_2$  are the densities and  $\mu_1$  and  $\mu_2$  are the viscosities. In all that follows  $g = 0$ . The cylinders at  $r = a$  and  $r = b$  rotate with some constant angular frequency  $\Omega$ . At the interface  $\Sigma$

$$F(x, r, \theta, t) = r - R(x, \theta, t) = 0, \quad (2.4)$$

we have

$$u = \frac{\partial R}{\partial t} + \frac{v}{R} \frac{\partial R}{\partial \theta} + w \frac{\partial R}{\partial x}. \quad (2.5)$$

We also require that the jumps across  $\Sigma$

$$[[\cdot]] = (\cdot)_1 - (\cdot)_2$$

in the velocity, the shear stress and the difference between the jump in the normal stress and the surface tension force all vanish.

We are going to study spatially periodic solutions which are  $2\pi/\alpha$  periodic in  $x$  and, of course,  $2\pi$  periodic in  $\theta$ . The volume of each component fluid is prescribed by specifying a mean radius

$$d^2 = \bar{R}^2 \quad (2.6)$$

where

$$\bar{(\cdot)} = \frac{\alpha}{4\pi^2} ((\cdot)), \quad (2.7)$$

$$((\cdot)) \stackrel{\text{def}}{=} \int_0^{2\pi/\alpha} dx \int_0^{2\pi} (\cdot) d\theta.$$

Our convention is the fluid with subscript 1 is on the inside. The fluid on the inside can be heavy or light. We are interested in two cases

(A) The heavy fluid inside,  $a \leq r \leq R_A(\theta, x, t)$

$$[[\rho]] > 0, \quad d_A^2 = \bar{R}_A^2 \quad (2.8)$$

(B) The heavy fluid is outside,  $R_B(\theta, x, t) \leq r \leq b$

$$[[\rho]] < 0, \quad d_B^2 = \bar{R}_B^2. \quad (2.9)$$

The volume ratio of heavy to light fluid is

$$m_A = \frac{d_A^2 - a^2}{b^2 - d_A^2}, \quad m_B = \frac{b^2 - d_B^2}{d_B^2 - a^2}. \quad (2.10)$$

The volume of light fluid and the volume of heavy fluid is fixed, independent

of whether it is inside or outside when  $m_A = m_B$ . Then

$$d_A^2 = \frac{m_b^2 + a^2}{1 + m}, \quad d_B^2 = \frac{b^2 + m_a^2}{1 + m}. \quad (2.11)$$

The jump equations at  $r = R$  are

$$[[\underline{u}]] = 0,$$

$$-([p]) + 2HT\underline{n} + [[\underline{S}]] \cdot \underline{n} = 0, \quad (2.12)$$

where

$$2H = \frac{RR_{\theta\theta}(1+R_x^2) + RR_{xx}(R^2+R_\theta^2) - R^2(1+R_x^2) - 2R_\theta^2 - 2RR_\theta R_x R_{x\theta}}{(R^2 + R_\theta^2 + R^2 R_x^2)^{3/2}} \quad (2.13)$$

$2H$  is the sum of the principal curvatures,  $T$  is the surface tension, and  $\underline{n} = \nabla F / |\nabla F|$  and  $\nabla F = \underline{e}_r - \underline{e}_\theta R_\theta - \underline{e}_x R_x$ .

### 3. RIGID ROTATION OF TWO FLUIDS

The velocity

$$\underline{u}_\theta = \underline{e}_\theta \Omega r, \quad (3.1)$$

and the pressure

$$p_\theta = \frac{\rho \Omega^2 r^2}{2} + D, \quad (3.2)$$

where the constants  $(\rho, D)$  are  $(\rho_1, D_1)$  in  $V_1$  and  $(\rho_2, D_2)$  in  $V_2$ , is a solution of (2.2), (2.3) and (2.4) with  $\underline{S}$  identically zero. We suppose that  $R(\theta, x, t)$ , periodic in  $x$  and  $\theta$ , is prescribed and arbitrary. At  $r=R$  we have

$$[[\underline{u}_0]] = 0,$$

(3.3)

$$[[p_0]] = [[\rho]] \frac{\Omega^2}{2} R^2 + [[D]].$$

We cannot satisfy the differential equation

$$[[p_0]] + 2HT = 0, \quad (3.4)$$

expressing the jump condition for the normal component of the stress, for an arbitrary given surface  $R(\theta, x, t)$ . We call (3.1) - (3.3) an "extended" rigid motion and we prove that these motions are globally stable with the shape of the interface, and possibly the placement of heavy and light liquid determined by a new minimum principle of classical type.

#### 4. PERTURBATION EQUATIONS

Let  $\underline{u}, p$  be the velocity and pressure in the deformed domain and let

$$\underline{u} = \underline{u}_0 + \underline{\hat{u}},$$

$$p = p_0 + \hat{p}$$

where  $\underline{u}_0 = \underline{e}_\theta V(r)$ , and  $p_0(r)$  is the associated pressure. The function  $\underline{\hat{u}}$ ,  $\hat{p}$  and  $\hat{\underline{S}}$  are perturbations. In designating components

$$\underline{\hat{u}} = (u, v, w)$$

we suppress the caret overbar. All these quantities are defined in  $V_1(t)$  and  $V_2(t)$ . For the moment we leave open the possibility that  $[[V]] \neq 0$ , (for rigid motion  $[[V]] = 0$ ). The equations governing  $\underline{\hat{u}}$  and  $\hat{p}$  are

$$\rho \left[ \frac{\partial \underline{\hat{u}}}{\partial t} + \underline{\hat{u}} \cdot \nabla \underline{u}_0 + \underline{u}_0 \cdot \nabla \underline{\hat{u}} + \underline{\hat{u}} \cdot \nabla \underline{\hat{u}} \right] = -\nabla \hat{p} + \nabla \cdot \hat{\underline{S}}, \quad (4.1)$$

$$\nabla \cdot \underline{\hat{u}} = 0. \quad (4.2)$$

The boundary conditions are

$$\hat{u}_1(r=a) = \hat{u}_2(r=b) = 0. \quad (4.3a,b)$$

The interface conditions on  $r = R(x, \theta, t)$  are

$$[[u]] = [[w]] = [[v+V]] = 0 \quad (4.4)$$

$$u = \frac{\partial R}{\partial t} + \frac{v+V}{R} \frac{\partial R}{\partial \theta} \quad (4.5)$$

$$-[[\hat{p}]]\underline{n} + [[\hat{S}]] \cdot \underline{n} = [[p_0]]\underline{n} + 2HT\underline{n} \quad (4.6)$$

## 5. ENERGY EQUATIONS FOR NONLINEAR DISTURBANCES

Introduce the following notations:

$$\langle \cdot \rangle = \int_V (\cdot) dV,$$

$$\langle \cdot \rangle_\Sigma = \int_\Sigma (\cdot) d\Sigma,$$

where

$$d\Sigma = R d\theta dx \sqrt{1 + R_\theta^2 / R^2 + R_x^2} = R d\theta dx |\nabla F|$$

and we are assuming that the disturbance flow is  $2\pi/a$  periodic in  $x$ .

To form the energy equation we multiply (4.1) by  $\underline{u}$ , integrate over  $V_1$  and  $V_2$ , add and use Reynolds transport theorem to show that

$$\begin{aligned} \frac{d}{dt} \left\langle \rho \frac{|\hat{\underline{u}}|^2}{2} \right\rangle &= \left\langle \rho \hat{\underline{u}} \cdot \frac{d\hat{\underline{u}}}{dt} \right\rangle \\ &= \left\langle \rho \hat{\underline{u}} \cdot \left[ \frac{\partial \hat{\underline{u}}}{\partial t} + (\hat{\underline{u}} + \underline{u}_0) \cdot \nabla \hat{\underline{u}} \right] \right\rangle \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} \left\langle \rho \frac{|\hat{\underline{u}}|^2}{2} \right\rangle &+ \left\langle \rho \hat{\underline{u}} \cdot \underline{D}[\hat{\underline{u}}_0] \cdot \hat{\underline{u}} \right\rangle + \left\langle 2\rho \underline{D}[\hat{\underline{u}}] : \underline{D}[\hat{\underline{u}}] \right\rangle \\ &= \left\langle [(-\hat{p}\underline{n} + \hat{\underline{S}} \cdot \underline{n})] \right\rangle_{\Sigma} \end{aligned} \quad (5.1)$$

For flow in circles with  $\underline{u}_0 = \underline{e}_\theta r \Omega$  we have

$$\left\langle \rho \hat{\underline{u}} \cdot \underline{D}[\underline{u}_0] \cdot \hat{\underline{u}} \right\rangle = \left\langle \rho u v r \frac{d\Omega}{dr} \right\rangle$$

which vanishes on rigid motions. Moreover, since  $\hat{\underline{u}} = \underline{u} - \underline{u}_0$  is continuous across  $\Sigma$ , we have

$$\begin{aligned} \left\langle [(\hat{\underline{u}} \cdot (-\hat{p}\underline{n} + \hat{\underline{S}} \cdot \underline{n}))] \right\rangle_{\Sigma} &= \left\langle (\underline{u} - \underline{u}_0) \cdot [(-\hat{p}\underline{n} + \hat{\underline{S}} \cdot \underline{n})] \right\rangle_{\Sigma} \\ &= \left\langle (\underline{u} - \underline{u}_0) \cdot \underline{n} ([p_0] + 2HT) \right\rangle_{\Sigma} \end{aligned} \quad (5.2)$$

where the last equality follows from (4.6). We next observe (see Joseph, II, Eq. (96.11)) that

$$\left\langle 2HT\underline{u} \cdot \underline{n} \right\rangle_{\Sigma} = -\frac{d}{dt} \int_{\Sigma} T d\Sigma. \quad (5.3)$$

Since  $T$  is constant, we have

$$\langle 2HT \underline{u} \cdot \underline{n} \rangle_{\Sigma} = -T \frac{d}{dt} ((\sqrt{R^2 + R_{\theta}^2 + R^2 K_x^2}))$$

where  $((\cdot))$  is defined in (2.7).

Moreover, using (2.5), we find that

$$\underline{u} \cdot \underline{n} = \underline{u} \cdot \frac{\nabla F}{|\nabla F|} = \frac{1}{|\nabla F|} \frac{\partial R}{\partial t}.$$

Hence

$$\langle [[p_0]] \underline{u} \cdot \underline{n} \rangle_{\Sigma} = (([[p_0]] R \frac{\partial R}{\partial t})).$$

Since  $[[p_0]] = [[\rho]] \Omega^2 R^2 / 2 + [[D]]$  is a function of  $R$  alone, there is a scalar function  $\phi(R)$ ,

$$\phi(R) = [[\rho]] \Omega^2 \frac{(R^2 - d^2)^2}{8}, \quad (5.4)$$

such that

$$(([[p_0]] R \frac{\partial R}{\partial t})) = \frac{d}{dt} ((\phi(R))); \quad (5.5)$$

in deriving (5.4) - (5.5), we used (2.6) to set

$$\frac{d}{dt} ((R^2)) = 0.$$

Finally, we show that, for rigid motions where  $\underline{u}_0 \cdot \underline{n} = \Omega R_{\theta} / |\nabla F|$ , we have

$$\begin{aligned} & \langle \underline{u}_0 \cdot \underline{n} ([[p_0]] + 2HT) \rangle_{\Sigma} \\ &= \Omega ((R R_{\theta} ([[p_0]] + 2HT))) = 0 \end{aligned} \quad (5.6)$$

In deriving the first equality, we use  $\underline{n} = \nabla F / |\nabla F|$ ; then we note that since  $[[p_0]]$  is a function of  $R$  alone, there is a  $2\pi$  periodic  $m(R)$  such that

$$R_\theta R[[p_0]] = \partial m(R) / \partial \theta$$

which vanishes on integration. We next use the expression (2.13) to write

$$RR_\theta 2H = \frac{\partial}{\partial \theta} \left[ \frac{-R(1+R_x^2)}{|\nabla F|} \right] + \frac{\partial}{\partial x} \left[ \frac{R_x R_\theta R}{|\nabla F|} \right].$$

Since  $R(\theta, x, t)$  is periodic in  $x$  and  $\theta$ , the last integral in (5.6) vanishes.

Collecting all these results, we find that

$$\frac{d(E+M)}{dt} = -D, \quad (5.7)$$

where

$$E = \left\langle \rho \frac{|\underline{u}|^2}{2} \right\rangle,$$

$$D = \left\langle 2\underline{u} \underline{D}[\underline{u}] : \underline{D}[\underline{u}] \right\rangle,$$

$$M = T((R^2 + R_\theta^2 + R_x^2)^{1/2}) - \frac{[[\rho]] \Omega^2}{8} ((R^2 - d^2)^2). \quad (5.8)$$

The function  $M[R]$  is the variable part of a "potential energy"  $P[R]$  for rigid motions. It is easily verified that



$$K_1 = [-(\langle a^* \rangle) + (\langle R^* \rangle)] \rho_1 \Omega^2 / 8,$$

$$K_2 = [(\langle b^* \rangle) - (\langle R^* \rangle)] \rho_2 \Omega^2 / 8$$

is the kinetic energy of a rigid motion in region  $V_1$  and  $V_2$ , respectively. The associated potential of this rigid motion is the negative of the kinetic energies plus the surface energy

$$\begin{aligned} P &\stackrel{\text{def}}{=} -(K_1 + K_2) + T \int_{\Sigma} d\Sigma \\ &= C_1 + T \left( \left[ R^2 (1 + R_x^2) + R_\theta^2 \right]^{1/2} \right) - [\rho] (\langle R^* \rangle) \Omega^2 / 8 \\ &= C_2 + M \end{aligned} \tag{5.9}$$

where  $C_1$  and  $C_2$  are different constants. We may write (5.7) as

$$\frac{d(E+P)}{dt} = -D. \tag{5.10}$$

Equations (5.7) and (5.10) were derived by D. D. Joseph.

In the next section we shall show that rigid motions are globally stable, as is the case with one fluid, but that the stable configurations of the rigid motions minimize  $P$  subject to the volume constraint (2.6).

It is useful to write the potential in a dimensionless form in which  $R = d + \delta$  and  $(\delta, x, 1/\alpha)$  are made dimensionless with  $d$  and  $\Delta = \delta/d$ . Then, to within constants, we may define a dimensionless potential

$$\bar{P} = -(\langle 1 \rangle) + \left( \left[ (1 + \Delta)^2 (1 + \Delta_x^2) + \Delta_\theta^2 \right]^{1/2} \right) + \frac{J}{8} (\langle 2\Delta + \Delta^2 \rangle), \tag{5.11}$$

where

$$J = - \frac{d^3[[\rho]]\Omega^2}{1}$$

has the sign of  $[[\rho]]$ . The constraint (2.6) implies that

$$((2\Delta + \Delta^2)) = 0. \quad (5.12)$$

This shows that the average deviation  $\Delta = -1 + R/d$  from zero must be negative if the volume of the two fluids is preserved.

It is necessary to remark that the representation  $r = R(\theta, x, t)$  of the free surface is not completely general and it loses its utility when the magnitude  $|\Delta|$  of the deviation from the cylinder is equal to

$$\bar{\Delta} = \min \left[ 1 - \frac{a}{d}, \frac{b}{d} - 1 \right]. \quad (5.13)$$

If  $|\Delta| \geq \bar{\Delta}$ , then the interface will touch one or the other of the cylinders, and a smooth free surface will not be possible.

The linearized form of (5.11) for  $\Delta$  near to zero is

$$\bar{P} = \frac{T}{2} ((J-1)\Delta^2 + \Delta_x^2 + \Delta_\theta^2). \quad (5.14)$$

## 6. STABILITY OF RIGID MOTION

The following results concern solutions of the equations which are smooth for all time.

**Theorem 1.** Rigid motions are stable in the sense that periodic disturbance of rigid motion must decay in the mean. We first note that  $D > 0$  on all non zero disturbances of rigid motion. We are considering the stability of flows with heavy fluid outside,  $-[[\rho]] = \rho_2 - \rho_1 > 0$ , or inside,  $[[\rho]] > 0$ . These two situations, called (A) and (B) in Section 2, are distinct in that they cannot be connected by time dependent motions with smooth interfaces. We choose  $\underline{u}$  to be a disturbance of one or the other of these two possibilities. Then the positivity of  $D$  implies the decay of  $E + P$ . In fact, we could show that there is  $\lambda$  depending on the viscosities and the densities, such that  $D \geq \lambda E$  where  $\lambda$  depends on  $v_1, v_2, \rho_1, \rho_2, \mu_1, \mu_2$ . Integrating (5.10) from  $t = 0$  to  $t$ ,

we find that

$$\begin{aligned} E(t) + P(t) &= E(0) + P(0) - \int_0^t D(\tau) d\tau \\ &\leq E_0 + P_0 - \lambda \int_0^t E(\tau) d\tau; \end{aligned} \quad (6.1)$$

it follows that

$$\lambda \int_0^t E(\tau) d\tau \leq E(0) + P(0) - E(t) - P(t). \quad (6.2)$$

Since  $P$  is bounded from below we conclude that

$$E(t) \text{ and } D(t) \text{ are integrable.} \quad (6.3)$$

Moreover, assuming that  $E$  goes to 0 as  $t \rightarrow +\infty$ ,  $P$  admits a finite limit as  $t$  goes to  $+\infty$ .

Let us consider the limit configuration  $(E(\infty), P(\infty))$  since  $E(\infty) = 0$ , this is a rigid motion. To show that  $P(\infty)$  is a minimum of the functional  $P$  as  $R$  varies we consider any rigid motion  $(E(0) = 0, P(0))$  and assume that this rigid motion goes to the rigid motion  $(E(\infty) = 0, P(\infty))$  as  $t$  goes to  $+\infty$ . If  $P(0) \neq P(\infty)$ , then the interface between the two liquids must have moved from the configuration at  $t = 0$  to the one at  $t = \infty$ , and in this motion  $D(t) > 0$  on an interval with non zero measure. Then, from (5.10)

$$P(\infty) - P(0) = - \int_0^{\infty} D(t) dt < 0, \quad (6.4)$$

so that

$$P(\infty) < P(0). \quad (6.5)$$

Thus  $P$  decreases in every change of configuration between rigid motions. Since  $P$  is a bounded from below functional of  $R$ ,  $P$  must decrease to

$$P(\infty) = \lim_{t \rightarrow \infty} P(t) = \min_R P(R), \quad (6.6)$$

where  $\bar{R}^2 = d^2$  and  $R$  is periodic and continuously differentiable in  $x$  and  $\theta$ .

We may describe the set of such periodic functions  $R$  with  $\bar{R}^2 = d^2$  as a one parameter family  $R(\epsilon)$ . We assume that  $R(0) = \bar{R}$  where

$$P(\bar{R}) = \min_R P(R) = \min_{\epsilon} P(R(\epsilon)). \quad (6.7)$$

We must have:

$$\frac{dP}{d\epsilon} [R(\epsilon)]_{\epsilon=0} = -((\bar{R} \frac{dR}{d\epsilon}(0)(2\bar{H}T + \frac{[[\rho]]}{2} \Omega^2 \bar{R}^2))) = 0, \quad (6.8)$$

where  $2\bar{H}$  denotes the curvature (2.13) evaluated on the minimizer  $R = \bar{R}$ . Since  $\bar{R}^2(\epsilon) = d^2$ , we have

$$((\bar{R} \frac{dR}{d\epsilon}(0))) = 0, \quad (6.9)$$

so that  $\bar{R} \frac{dR}{d\epsilon}(0)$  is orthogonal to constants. It follows now from (6.8) and (6.9) that

$$2\bar{H}T + [[\rho]] \frac{\Omega^2 \bar{R}^2}{2} = C = \bar{2H}T + [[\rho]] \frac{\Omega^2 d^2}{2}, \quad (6.10)$$

where  $\bar{H}$  is the mean value of  $\bar{H}$ .

Equation (6.10) may be recognized as the differential equation arising from the normal stress condition (2.12) on rigid motions.

It now follows that extended rigid motions, which are globally stable, are actually hydrodynamically admissible, with a balanced normal stress equa-

tion, when the interface  $R = \bar{R}$  is a minimizer of (6.6). The stable rigid motions satisfy (2.5) in the form

$$0 = \frac{\partial \bar{R}}{\partial t} + \Omega \frac{\partial \bar{R}}{\partial \theta}.$$

where  $\bar{R}$  is time independent in a rotating coordinate system.

Theorem 2. The stable configurations are those rigid motions which minimize  $P$  among  $C^1(x, \theta)$  functions  $R(x, \theta)$  satisfying the volume constraint (2.6).

Consider first case (A) in which the heavy fluid is outside,  $J > 0$ . If  $J > 1$ , then  $\bar{P}$  given by (5.14) is a minimum when  $\Delta = 0$ .

Theorem 3. The cylindrical interface with constant radius  $R = d$ , is stable against small disturbances if and only if  $J \geq 1$ .

It is of interest to ask when  $R = d$  is a global minimum of  $P$  among all interfaces of the form  $r = R(\theta, z)$  compatible with the volume constraint. This question is answered by the following theorem of M. Renardy.

Theorem 4. The concentric interface  $R = d$  is a global minimum of  $P$  among all interfaces  $r = R(\theta, z)$  satisfying (3.4) if and only if  $J \geq \frac{4}{\left[1 + \frac{a}{d}\right]^2}$ .

First we show that the criterion of theorem 4 is sufficient for stability. Certainly we have  $P \geq \hat{P} = TR + \frac{1}{8} \Omega^2 (\rho_2 - \rho_1) R^4$ , and it is sufficient to show that  $R = d$  minimizes  $\hat{P}$ . We set  $R^2 = d^2(1 + \gamma)$ , hence  $\gamma$  is subject to the constraints  $-1 + \frac{a^2}{d^2} \leq \gamma \leq -1 + \frac{b^2}{d^2}$  and  $((\gamma)) = 0$ . We then have

$$\begin{aligned} \hat{P} &= Td \left( (\sqrt{1+\gamma} + \frac{1}{8} J (1+\gamma)^2) \right. \\ &\quad \left. - Td \left( (\sqrt{1+\gamma} + \frac{1}{8} J (1+\gamma)^2 - \gamma \left[ \frac{1}{2} + \frac{1}{4} J \right]) \right) \right). \end{aligned} \quad (6.11)$$

We define

$$f(\gamma) = \sqrt{1+\gamma} + \frac{1}{8} J (1+\gamma)^2 - \gamma \left[ \frac{1}{2} + \frac{1}{4} J \right]. \quad (6.12)$$

The constant multiplying  $\gamma$  has been chosen such that  $f'(0) = 0$ . If  $J \geq$

$$\frac{4}{\left[1 + \frac{a}{d}\right]^2}, \text{ then } f \text{ has its minimum at } \gamma = 0, \text{ for } \gamma \text{ in the range}$$

$$\left[-1 + \frac{a^2}{d^2}, -1 + \frac{b^2}{d^2}\right].$$

The criterion is also necessary for a global minimum. We can choose  $R$  such that  $R_0 = 0$ , and, by choosing a small (long waves) we can make  $R_2$  as small as we like. It follows that it is also necessary that  $R = d$  minimizes  $\hat{P}$ . If  $J < \frac{4}{\left[1 + \frac{a}{d}\right]^2}$ , then  $f\left[-1 + \frac{a^2}{d^2}\right] < f(0)$ , and the graph of  $f$  is sketched as

Figure 1. If we draw the tangent from the point  $\left(-1 + \frac{a^2}{d^2}, f\left(-1 + \frac{a^2}{d^2}\right)\right)$  as indicated, it will touch the graph at a point  $\alpha, f(\alpha)$  to the right of  $\gamma = 0$ , and it will intersect the line  $\gamma = 0$  at a value below  $f(0)$ . This means that we can take a perturbation  $\gamma$ , alternating between the values  $-1 + \frac{a^2}{d^2}$  and  $\alpha$ , such that  $\langle \gamma \rangle = 0$ , but  $\langle f(\gamma) \rangle < f(0)$ .

Turning next to case B in which the heavy fluid is inside  $J < 0$  we find that  $f(\gamma)$  is concave. Hence  $\hat{P}$  will have minimizers only at boundary values  $-1 + \frac{a^2}{d^2}$  and  $-1 + \frac{b^2}{d^2}$ . In an infinite cylinder, there will therefore be no minimizers of  $P$  of the form  $r = R(\theta, z)$ . In a finite cylinder, we cannot make  $R_2$  arbitrarily small without also making  $\gamma$  small, and there can be stable motions with heavy fluid inside, which have a corrugated free surface, as in the experiments of Yih (1960) and Moffatt (1977). It would be of interest to determine these corrugated shapes as a solution for the minimum problem of  $P$ . If  $J$  is large, we expect the amplitude of the corrugated surface to be also large and eventually violate the constraints  $-1 + \frac{a^2}{d^2} < \gamma < -1 + \frac{b^2}{d^2}$ .

The results of this paper have some relevance for the problem of centrifuging. Intuitively one expects that the heavy fluid will be outside if the rate  $\Omega$  of rotation is large, even if the heavy fluid were initially on the inner cylinder. The transport of fluid from the inner to the outer cylinder is a topologically complex process which cannot be handled in the frame of our smooth parameterization of the interface. The transport of fluid from the inner to the outer surface is also a physically complex process involving the

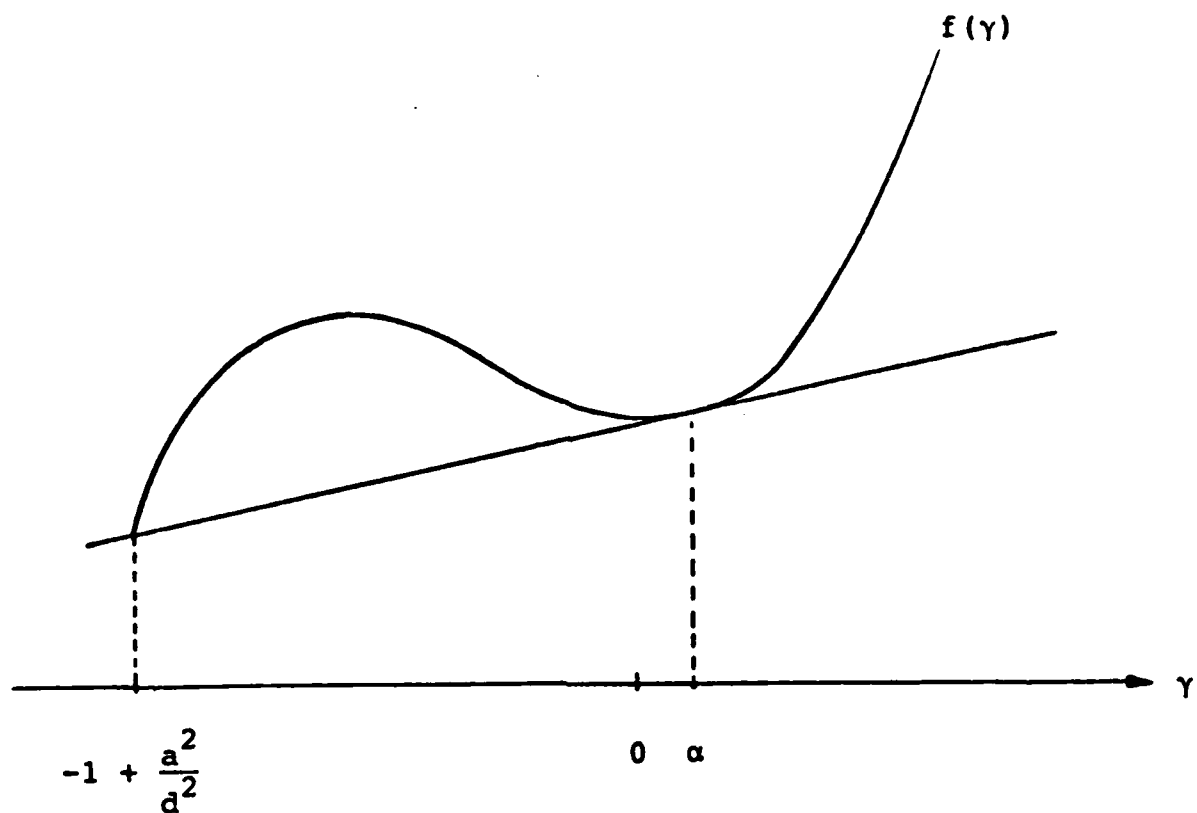


Figure 1

rupture of adhesion at both walls and possibly internal fracturing and healing of the liquids themselves. These physical processes are not well understood, and they do not appear in our equations. Nevertheless, it is not unreasonable to seek stable configurations among those which minimize  $P$  with respect also to the position of heavy and light fluid.

We can compare the potentials  $P_A$  and  $P_B$  for flow with heavy fluid inside  $[[\rho]] > 0$  and heavy fluid outside  $[[\rho]] < 0$  under the condition that the volume  $m$ , defined in Section 2, is fixed. The total kinetic energy  $K_A$  when the heavy fluid with density  $\rho_1$  is inside is given by

$$K_A = K_1 + K_2 = \frac{\rho_1 \Omega^2}{8} ((R_A^2 - a^2)) + \frac{\rho_2 \Omega^2}{8} ((b^2 - R_A^2))$$

where  $((R_A^2)) = ((d_A^2))$ . When the heavy fluid with density  $\rho_1$  is outside

$$K_B = K_1 + K_2 = \frac{\rho_1 \Omega^2}{8} ((b^2 - R_B^2)) + \frac{\rho_2 \Omega^2}{2} ((R_B^2 - a^2))$$

where  $((R_B^2)) = ((d_B^2))$ . Using (2.10), with  $m_A = m_B = m$  we find that

$$((R_A^2)) = \frac{mb^2 + a^2}{1+m}, \quad ((R_B^2)) = \frac{b^2 + ma}{1+m}$$

The potentials are

$$P_A = -K_A + T |\Sigma_A|,$$

$$P_B = -K_B + T |\Sigma_B|$$

where

$$|\Sigma_A| = \int_{\Sigma_A} d\Sigma$$



is the area of  $\Sigma_A$ . The difference in the potentials is

$$P_A - P_B = -K_A + K_B + T(|\Sigma_A| - |\Sigma_B|) \\ = \frac{[[\rho]]\Omega^2}{8} ((b^2 + a^2 - R_A^2 - R_B^2)) + T(|\Sigma_A| - |\Sigma_B|)$$

Here  $[[\rho]] = \rho_1 - \rho_2 > 0$ . If  $P_A > P_B$  then (B) with heavy fluid outside is more stable. Since  $((R^2)) = ((d^2))$ , we have

$$((R^2 - d^2)) = ((R^2)) - ((d^2)).$$

The potential difference may be written as

$$P_A - P_B = \frac{[[\rho]]\Omega^2}{8} \left[ ((b^2 + a^2 - d_A^2 - d_B^2)) - (([R_A^2 - d_A^2]^2 + [R_B^2 - d_B^2]^2)) \right] + T(|\Sigma_A| - |\Sigma_B|) \quad (6.1)$$

Now, using (2.11), we eliminate  $d_A^2$  and  $d_B^2$

$$P_A - P_B = \frac{[[\rho]]\Omega^2}{8} \left[ \frac{2m((b^2 - a^2)^2)}{(1+m)^2} - (([R_A^2 - d_A^2]^2 + [R_B^2 - d_B^2]^2)) \right] + T|\Sigma_A| - T|\Sigma_B| \quad (6.2)$$

Consider now the case of uncorrugated interfaces with  $R_A = d_A$ ,  $R_B = d_B$ ,  $\Sigma_A = 2\pi d_A$ ,  $\Sigma_B = 2\pi d_B$ . We find that

$$P_A - P_B = \frac{[[\rho]]\Omega^2}{4} m \frac{((b^2 - a^2)^2)}{(1+m)^2} - 2\pi T((d_A - d_B)) \quad (6.3)$$

where  $d_A$  and  $d_B$  are given in terms of  $m$  by (2.11). If the volume ratio  $m$  of heavy ( $\rho_1$ ) to light ( $\rho_2$ ) fluid is greater than one, then  $d_A > d_B$ . Hence  $P_A > P_B$  when  $m > 1$ . If there is only a small amount of heavy fluid,  $m < 1$  and  $d_B < d_A$  then  $P_A > P_B$  if  $[[\rho]]\Omega^2/T$  is large. In all these cases the configuration with

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## 7. STABILITY OF ROLLERS

Rollers are viscous fluid bodies which rotate as rigid wheels in fluids of smaller viscosity. These rollers have been observed (Joseph, Nguyen and Beavers, 1983) in bicomponent flows of immiscible liquids in several different flow configurations: on a cylinder rotating in a box, between the four cylinders of Taylors mill for studying straining flows, and separating dynamically driven Taylor vortices between rotating cylinders.

The most interesting feature of the dynamics leading to the formation of rollers is the fracturing of the viscous liquid at some critical level of the stress. In this process the roller breaks away from the side wall and relieves the high stress associated with no slip at the side wall. So in the final, stable dynamics, rollers are lubricated by water and air on all sides. The rollers rotate nearly as rigid bodies because they are so viscous. The stability of rollers, as our analysis suggests, depends on the fact that the density stratification is such as to prevent the centrifuging of the roller. The viscosity ratio is probably not an important factor in the dynamics of stable rollers.

The water which surrounds the rollers in experiments is at rest near the tank wall and cannot rotate rigidly. Therefore rollers are not a special case of rigid motions studied in this paper. However the density stratification, with water outside, does contribute to the stability of rollers, with a stabilizing term  $[[\rho V^2]]\delta^2$  at the interface, where  $V$  is the common velocity of fluid particles on either side of the interface and  $\delta$  is the surface deflection; here assumed small.

Rollers are unstable to non-axisymmetric disturbances when the angular velocity is high enough. This instability is associated with viscous shearing,

which becomes important at higher speeds and with a possible unstable distribution of angular momentum.

The low speed rollers are robustly stable. In our analysis we did not consider gravity, but gravity does enter into the dynamics of the stable rollers reported in the paper of Joseph, Nguyen and Beavers (1984) and here. In experiments in which the top of the roller rotates in air the roller would centrifuge out into the air were it not for gravity which on the small top portion of the roller exposed to air, is nearly radial. A similar, but smaller effect due to gravity occurs at the bottom of the roller which is pushed up by gravity because the lighter oil is buoyant in water. Gravity tends to flatten rollers into right circular cylinders. To a degree the diameter of stable rollers can be controlled by gravity, with a tendency for the roller to poke its head into the air. We are able to change the diameter of the rollers by changing the water level in the box. This effect of gravity is exhibited in plates 7.1 a-c. Sketches of the side view of these plates is shown in Figure 7.1 a-c.

The principal effect of gravity may be eliminated by submerging the roller entirely in water, as in Plates 7.2(a) and (b). When the flattening effects of gravity are absent the shape of the interface on stable rollers is strongly influenced by interfacial tension, with bounding surfaces in nearly circular arcs, as in Figure 7.2(b). The pressure distribution in the water is not a strong barrier to centrifuging and the dynamics of the roller in Plate 7.2 are closer to case B of this paper in which the heavy fluid is on the inside cylinder with a corrugated interface separating the two liquids.

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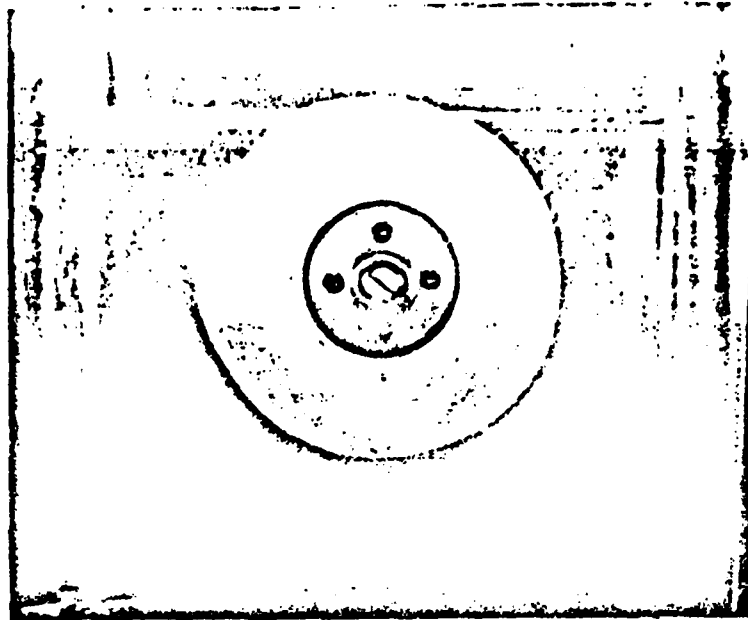


Plate  
7.1(a)

Plate 7.1: Roller of silicone oil ( $\rho = 0.95 \text{ gm/cc}$ ,  $\mu = 95,000 \text{ cp}$ ) in water at different water levels. The rod is made of plexiglass, 2 inches in diameter, rotates at 10 RPM.

(a) The roller is very nearly in a solid body rotation with small shearing by water at the roller rim. Part of the roller is in water and the other in air. The roller is very stable, held together by hydrostatic pressure in water and gravity in air.

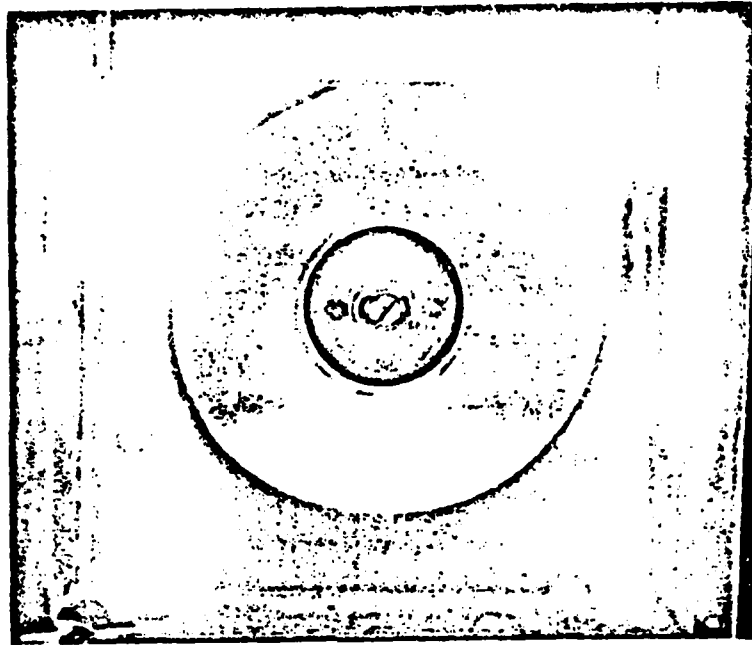


Plate  
7.1(b)

(b) Water is added to the box. The roller becomes larger by flattening out but remains round and stable.

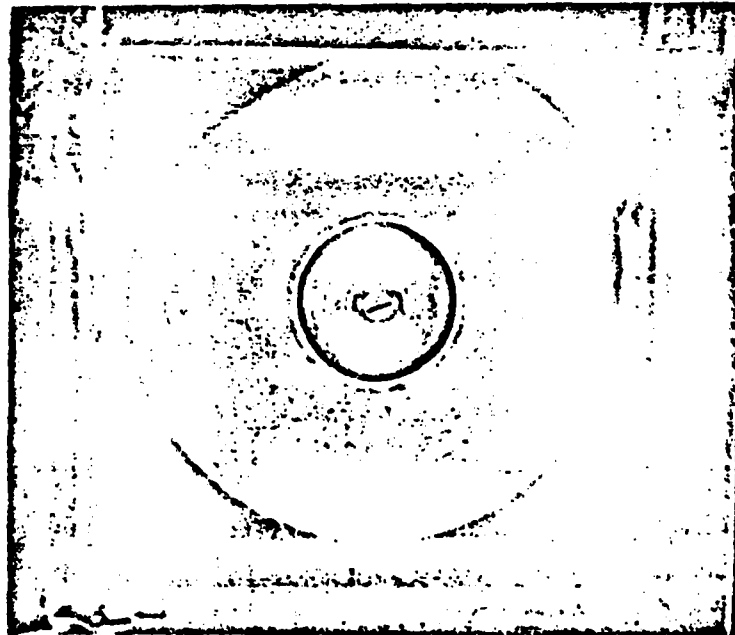


Plate  
7.1(c)

(c) More water is added. The roller becomes even larger. The roller is now completely submerged in water and is slightly out of round due to buoyancy.

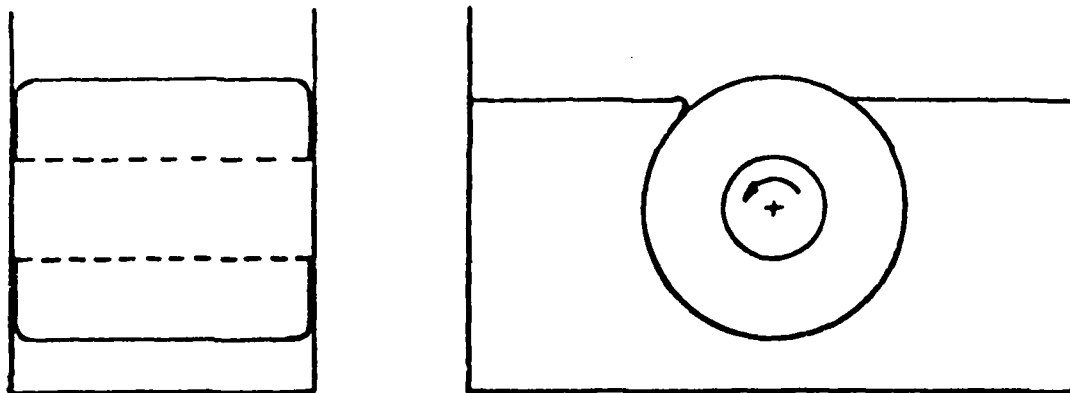


Figure 7.1(a)

Figure 7.1(a): Sketch of the side view and front view of plate 7.1(a).

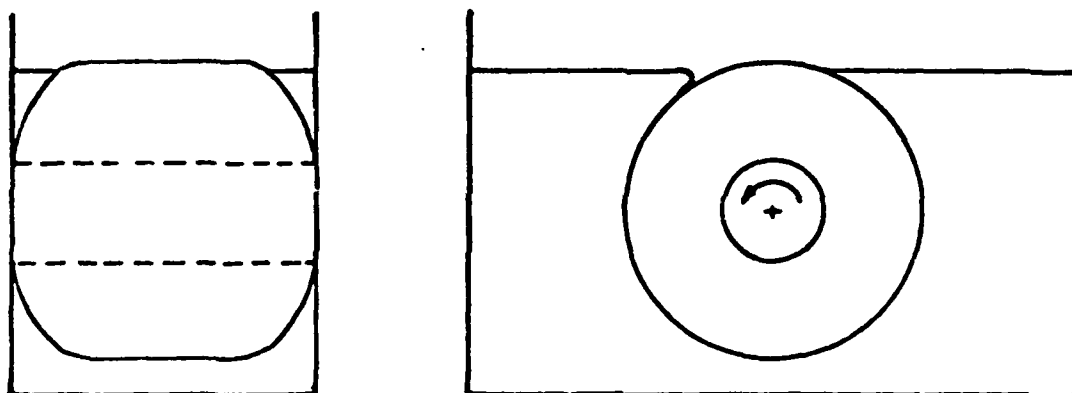


Figure 7.1(b)

Figure 7.1(b): Sketch of the side view and front view of Plate 7.1(b). Water is added to the box. The diameter of the roller becomes larger. The shape of the roller changes, conserving volume.

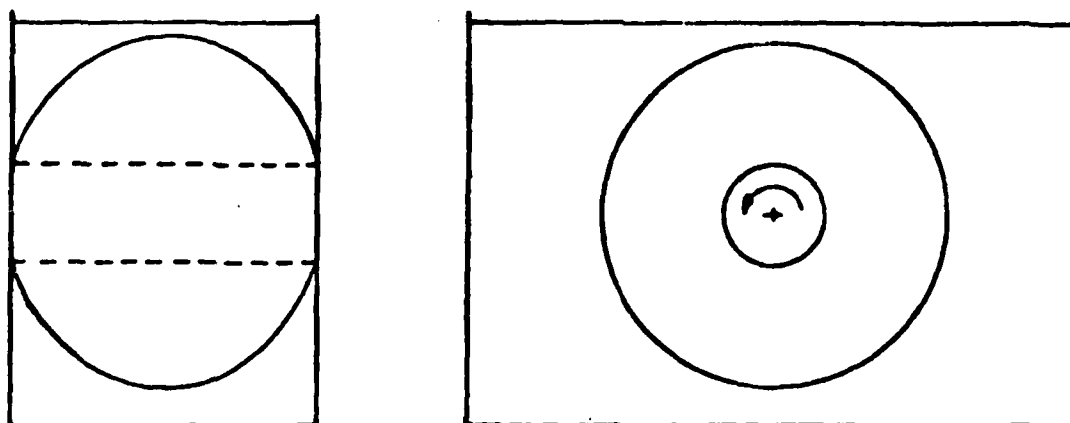


Figure 7.1(c)

Figure 7.1(c): Sketch corresponding to Plate 7.1(c).

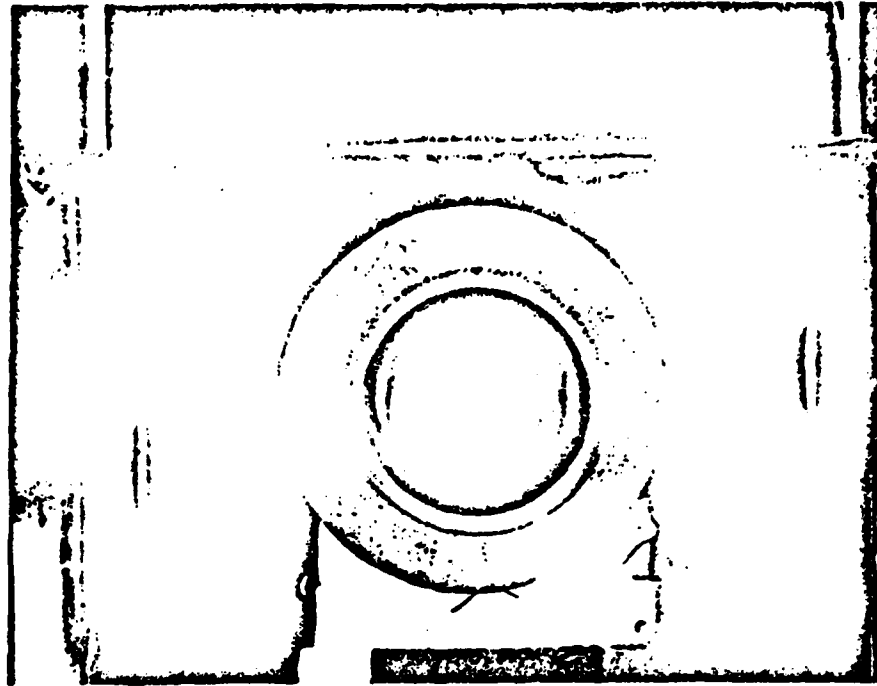


Plate  
7.2(a)

Plate 7.2(a): Front view of a completely submerged roller rotating at about 1.5 RPM.

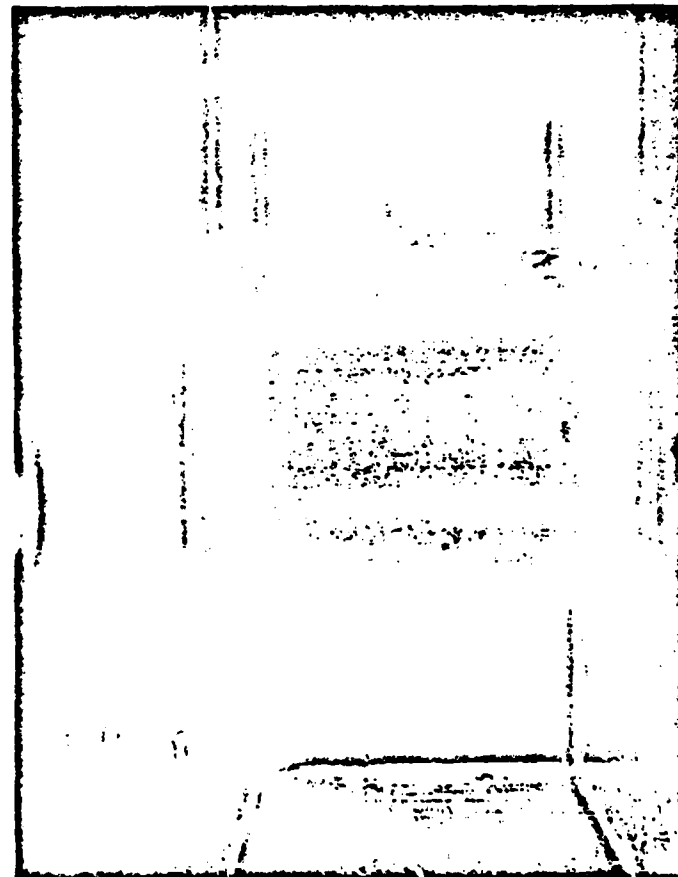


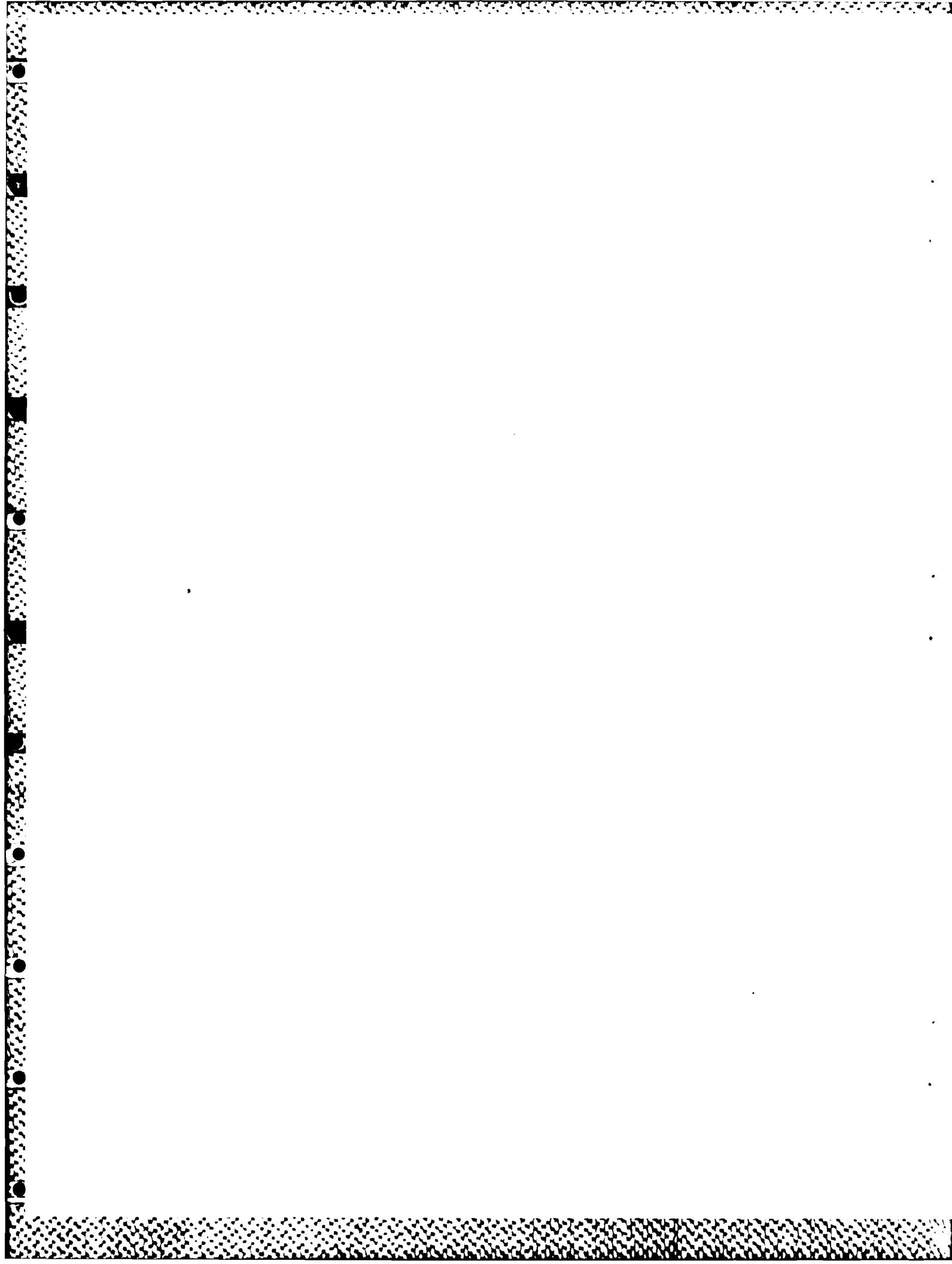
Plate  
7.2(b)

Plate 7.2(b): Side view of the submerged roller of Plate 7.2(a).

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20. ABSTRACT (cont.)

terface which separates the two fluids is determined by a minimizing problem for a potential  $P$  defined as the negative of the sum of the kinetic energies of two rigid motions plus the surface energy of the interface. We show that the stable interface between fluids has a constant radius when heavy fluid is outside and  $J$  is larger than one, where  $J = -d^3[[\rho]]\Omega^2/T$  where  $d$  is the mean radius,  $[[\rho]] < 0$  the density difference and  $T$  the surface tension. When  $J$  is negative the heavy fluid is inside and the interface must be corrugated. The potential of flows with heavy fluid outside is smaller, thus relatively more stable, than when light fluid is outside, whenever  $J$  is large or for any  $J$  when the volume ratio  $m$  of heavy to light fluid is greater than one. These results give partial explanation of the stability and shape of rollers of viscous oils rotating in water and the corrugation of the free surface of films coating rotating cylinders.

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